(Part 1). §11.6 Ratio Test for positive series

- **Ratio Test**: Give $\sum_{n=1}^{\infty} a_n$, where a_n are all positive. Consider $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, we have the following three cases:

$$\begin{cases} \text{(i)} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1, & \text{series } \sum a_n \text{ is convergent.} \\ \text{(ii)} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1, & \text{series } \sum a_n \text{ is divergent.} \\ \text{(iii)} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L = 1, & \text{(ratio test) is } \boxed{\text{inconclusive.}} \end{cases}$$

Motivation from Geometric Series:

Suppose $\frac{a_{n+1}}{a_n} = L$ (without limit). Then $a_n = a \cdot L^{n-1}$ is a Geometric Sequence ($a_1 = a$)
and L is the common ratio (r). We have $L \geq 1$, DIV and $|L| < 1$ CONV.

Remark: (i) $L < 1$ includes the case $L = 0$, i.e., $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$

(ii) $L > 1$ includes the case $L = \infty$, i.e., $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$

(iii) Do not apply Ratio Test to p-Series or (p-Series like). You will get limit 1.

(Vi) Ratio Test should be applied for a_n with n-factorial

$$\boxed{n! = 1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n}.$$

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e.g. 1. Test $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ for convergence via ratio test.

Step 0: (compute $\frac{a_{n+1}}{a_n}$) $a_n = \frac{n^2}{3^n}$, $a_{n+1} = \frac{(n+1)^2}{3^{n+1}}$, $\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}} = \frac{(n+1)^2 \cdot 3^n}{3^{n+1} \cdot n^2}$

(combine 'Similar parts') $= \frac{(n+1)^2}{n^2} \cdot \frac{3^n}{3^{n+1}}$

Step 1: (compute $\lim \frac{a_{n+1}}{a_n}$) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{3^n}{3^{n+1}} = \left[\lim_{n\to\infty} \frac{(n+1)^2}{n^2}\right] \cdot \frac{1}{3}$.

i.e. $L = \frac{1}{3} < 1$.

$= 1 \cdot \frac{1}{3}$ (The limit is 1 according to leading term rule).

Step 2: $L < 1$. (Draw conclusion).

$\sum a_n = \sum \frac{n^2}{3^n}$ is convergent because of Ratio Test.

• eg 2. $\sum_{n=0}^{\infty} \frac{4^n}{n!\,3^n}$

(S15, MC) (Step 0:) $a_n = \frac{4^n}{n!\,3^n}$, $a_{n+1} = \frac{4^{n+1}}{(n+1)!\,3^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{4^{n+1}}{(n+1)!\,3^{n+1}}}{\frac{4^n}{n!\,3^n}} = \frac{4^{n+1}}{(n+1)!\,3^{n+1}} \cdot \frac{n!\,3^n}{4^n} = \boxed{\frac{4^{n+1}}{4^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{3^n}{3^{n+1}}}$$ (combine similar part) and cancel out

Hint: $n! = 1\times2\times3\times\cdots\times(n-2)\times(n-1)\times n$ $\qquad = \boxed{4 \cdot \frac{1}{n+1} \cdot \frac{1}{3}}$

$(n+1)! = 1\times2\times3\times\cdots\times(n-2)\times(n-1)\times n\times(n+1)$

(Step 1) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4}{3}\cdot\frac{1}{n+1} = 0 < 1$ (special case in (i))

$(L = 0 < 1)$

Step 2: $\sum_{n=1}^{\infty} \frac{4^n}{n!\,3^n}$ is convergent according to Ratio Test since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L = 0 < 1$

• eg 3: $\sum_{n=1}^{\infty} n!\,e^{-n}$, $a_n = n!\,e^{-n}$, $a_{n+1} = (n+1)!\cdot e^{-(n+1)}$

(F15, MC)

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\cdot e^{-(n+1)}}{n!\cdot e^{-n}} = \frac{(n+1)!}{n!}\cdot\frac{e^{-(n+1)}}{e^{-n}} = (n+1)\cdot e^{-1}$$

Hint: $\boxed{(n+1)! = n!\times(n+1)}$

$\frac{e^{-(n+1)}}{e^{-n}} = e^{-(n+1)}\cdot e^{+n} = e^{-(n+1)+n} = \boxed{e^{-1}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (n+1)\cdot e^{-1} = \infty > 1$ $(L = \infty > 1)$, Therefore, $\sum n!\,e^{-n}$ is divergent.

• eg 4. $\sum_{n=1}^{\infty} \frac{2n\cdot(n+1)}{n!}$, $a_n = \frac{2n\cdot(n+1)}{n!}$, $\boxed{a_{n+1} = \frac{2(n+1)\cdot(n+1+1)}{(n+1)!}}$ (replace n by $n+1$ in a_n to get a_{n+1})

(f14, 12pts).

$$\frac{a_{n+1}}{a_n} = \frac{2(n+1)\cdot(n+2)}{(n+1)!}\cdot\frac{n!}{2n\cdot(n+1)} = \frac{\cancel{2(n+1)}\cdot(n+2)}{\cancel{2}\,n\cdot\cancel{(n+1)}}\cdot\frac{n!}{(n+1)!}$$

$$= \frac{n+2}{n}\cdot\frac{1}{n+1} = \frac{n+2}{n^2+n}.$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+\cancel{2}}{n^2+\cancel{n}} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0 < 1, \ (L = 0 < 1)$$

According to Ratio Test, $\sum a_n = \sum \frac{2n\cdot(n+1)}{n!}$ is CONV since $L < 1$.

§11.5 Alternating Series. (A.S.)

- Notation: $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot b_n = b_1 - b_2 + b_3 - b_4 + \cdots$, where all $b_n > 0$ (positive), is called an alternating series.
- Remark 1: One key feature of A.S. is that POSITIVE and NEGATIVE terms appear alternatively. Therefore, $(-1)^{n-1}$ can be replaced by $(-1)^n$, $(-1)^{n+1}$ etc, i.e., $\sum(-1)^n\cdot b_n$, $\sum(-1)^{n+1}b_n$ are all A.S.

- Alternating Series Test: If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot b_n$ satisfies:
(A.S.T.) (i) $0 < b_{n+1} \le b_n$ for all n. (ii) $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

- e.g. 0. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$. Hint: $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n} = -\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}\cdots$, alternating with $b_n=\frac{1}{n}$.

Consider A.S. Test for $b_n = \frac{1}{n}$. (i) $0<\frac{1}{n+1}\le\frac{1}{n}$ for all n (ii) $\lim_{n\to\infty}\frac{1}{n}=0$

Therefore, $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n}$ is convergent according to A.S. Test.

- Remark 2: $\underset{(\text{A.S.})}{\sum_{n=1}^{\infty}(-1)^n\cdot\frac{1}{n}}$ and $\underset{(p\text{-Series}, p=1)}{\sum_{n=1}^{\infty}\frac{1}{n}}$ are two DIFFERENT series. The first CONV and the second DIV.

- e.g. 1. $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{2n+3}{3n+1}$. Alternating S. $b_n = \frac{2n+3}{3n+1}$, $\lim_{n\to\infty}\frac{2n+3}{3n+1} = \frac{2}{3} \neq 0$

According to $\boxed{\text{nth term test for DIV}}$, $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{2n+3}{3n+1}$ is divergent. (A.S.T. is inconclusive and not necessary)

- Remark 3: Flow chart to test Alternating Series: $\sum(-1)^n\cdot b_n$.

$\boxed{\text{① Find } b_n} \Longrightarrow \boxed{\text{② Compute } \lim_{n\to\infty} b_n}$ ⟹ $\boxed{\text{③.1 limit} \neq 0,\ \sum(-1)^n b_n \text{ DIV (DIV Test). Done}}$

⟹ $\boxed{\text{③.2 limit} = 0,\ \text{(ii) is satisfied}} \Rightarrow \boxed{\text{④ 'check' } b_n \text{ is decreasing}} \Downarrow \boxed{\text{⑤ conclusion. A.S. is CONV.}}$

- e.g. 2. $\sum_{n=1}^{\infty}\frac{(n+1)^2}{n^3}\cdot(-1)^n$

Solution: $b_n = \frac{(n+1)^2}{n^3}$, $\lim_{n\to\infty}\frac{(n+1)^2}{n^3} = \lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^3}} = \frac{1}{\infty} = 0$. and $\boxed{b_n \text{ is decreasing}}$ (do not need to check).

According to A.S.T., $\sum_{n=1}^{\infty}\frac{(n+1)^2}{n^3}\cdot(-1)^n$ is convergent.

- e.g. 3. Find the sum of $\sum_{n=0}^{\infty}\frac{4\cdot(-1)^n(3)^n}{5^n}$. Remark: This is AN A.S., but also a GEOMETRIC Series.
(S15, 14pts)

$a_n = \frac{4\cdot(-1)^n\cdot(3)^n}{5^n} = 4\cdot\left(\frac{-3}{5}\right)^n$, $n=0,1,2,\cdots$, G.S: $a=4$, $\boxed{r=\frac{-3}{5}}$

G.S. formula: $\sum_{n=0}^{\infty}a_n = \frac{a}{1-r} = 4\cdot\frac{1}{1-[-\frac{3}{5}]} = 4\cdot\frac{1}{1+\frac{3}{5}} = 4\cdot\frac{1}{\frac{8}{5}} = 4\cdot\frac{5}{8} = \boxed{\frac{5}{2}}$

§11.6 Absolute Convergence and the Ratio Test (Part 2).

- Absolute Convergence: We say $\sum a_n$ is ABSOLUTELY convergent if $\sum |a_n|$ is convergent. (Abs. Conv).

Remark 1: In particularly, for an alternating series $\sum (-1)^n \cdot b_n$, it is ABS conv if $\sum b_n$ is convergent.

Remark 2: ABS conv $\Rightarrow$ conv, i.e., if $\sum a_n$ is ABS conv, then it is conv. (You do not need to check conv of $\sum a_n$ anymore).

e.g. 1. $\sum (-1)^n \cdot \frac{1}{n}$ is conv but NOT ABS CONV because:

(1). According to A.S. Test (e.g. 0 in §11.5), $\sum (-1)^n \cdot \frac{1}{n}$ is conv.

(2). $\sum \left| (-1)^n \cdot \frac{1}{n} \right| = \sum \frac{1}{n}$ is NOT [redacted] conv, we say $\sum (-1)^n \cdot \frac{1}{n}$ is NOT ABS CONV.

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* eg 2 (S16, MC) Which statement is true for $\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k-3}$, is I conv and II Abs conv.

Solution: check ABS conv first. According to the definition of ABS conv, we say $\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k-3}$ is ABS conv if $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{2^k-3} \right| = \sum_{k=1}^{\infty} \frac{1}{2^k-3}$ is convergent.

Therefore, it is enough to check whether $\sum_{k=1}^{\infty} \frac{1}{2^k-3}$ conv or DIV. $\sum_{k=1}^{\infty} \frac{1}{2^k-3}$ is conv because of LIMIT Comparison Test. (Hint: $a_k = \frac{1}{2^k-3}$, $b_k = \frac{1}{2^k}$. Then $\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1}{2^k-3}}{\frac{1}{2^k}}$
$= \lim_{k\to\infty} \frac{2^k}{2^k-3} = \lim_{k\to\infty} \frac{1}{\frac{2^k-3}{2^k}} = \lim_{k\to\infty} \frac{1}{1-\frac{3}{2^k}} = \frac{1}{1-\frac{3}{\infty}} = \frac{1}{1-0} = 1$. And $\sum b_k$ is conv. Therefore, $\sum a_k = \sum \frac{1}{2^k-3}$ is conv.)

Conclusion: Since $\sum_{k=1}^{\infty} \left| \frac{(-1)^k}{2^k-3} \right| = \sum_{k=1}^{\infty} \frac{1}{2^k-3}$ is conv, $\sum \frac{(-1)^k}{2^k-3}$ is ABS conv and also conv. Both I and II are true.

- (The full version) Ratio Test (for ABS conv). Consider series: $\sum a_n$.

(i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\sum a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive.

Remark: Compare the Full Version with the Positive Term Version in Part 1 of §11.6.

• e.g. 3. Determine whether $\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k-3}$ is $\overbrace{\text{ABS}}$ conV or DIV by Ratio Test.

Solution: $a_k = \frac{(-1)^k}{2^k-3}$, $a_{k+1} = \frac{(-1)^{k+1}}{2^{k+1}-3}$. Hint: $(-1)^n = 1$ or -1, therefore $|(-1)^n| = 1$ for all n.

$$|a_k| = \frac{1}{2^k-3}, \quad |a_{k+1}| = \frac{1}{2^{k+1}-3}, \quad \left|\frac{a_{k+1}}{a_k}\right| = \frac{|a_{k+1}|}{|a_k|} = \frac{\frac{1}{2^{k+1}-3}}{\frac{1}{2^k-3}} = \frac{2^k-3}{2^{k+1}-3}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{2^k-3}{2^{k+1}-3} \overset{\text{l'Hosp}}{=\!=} \lim_{k\to\infty}\frac{\ln 2\cdot 2^k}{\ln 2\cdot 2^{k+1}} = \frac{1}{2} = L < 1.$$

Therefore, $\sum a_k = \sum \frac{(-1)^k}{2^k-3}$ is ABS conV due to Ratio Test.

• e.g. 4. (ww1, Ratio Test (iii), inconclusive)

Determine the convergence and ABS convergence of $\sum_{n=1}^{\infty} (-1)^n\cdot\frac{2n}{n^3+8}$

Solution: ABS conV: (Ratio Test) $a_n = (-1)^n\cdot\frac{2n}{n^3+8}$, $a_{n+1} = (-1)^{n+1}\cdot\frac{2(n+1)}{(n+1)^3+8}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{2(n+1)}{(n+1)^3+8}}{\frac{2n}{n^3+8}} = \frac{2(n+1)}{(n+1)^3+8}\cdot\frac{n^3+8}{2n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2(n+1)}{(n+1)^3+8}\cdot\frac{n^3+8}{2n} = \lim_{n\to\infty}\frac{(n+1)\cdot(n^3+8)}{[(n+1)^3+8]\cdot n} \underset{\text{terms}}{\overset{\text{leading}}{=\!=}} \lim_{n\to\infty}\frac{n\cdot n^3}{n^3\cdot n} = 1$$

The ratio test is inconclusive. (The ratio test does not tell us whether it is ABS conV or NOT)

Definition of ABS conV: $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{2n}{n^3+8}$ is ABS conV if $\sum_{n=1}^{\infty}\frac{2n}{n^3+8}$ is conV.

Consider $\frac{2n}{n^3+8}$ and $b_n = \frac{2n}{n^3} = \frac{2}{n^2}$, then $\lim_{n\to\infty}\frac{\frac{2n}{n^3+8}}{\frac{2}{n^2}} = \lim_{n\to\infty}\frac{2n}{n^3+8}\cdot\frac{n^2}{2} = 1$

and $\sum b_n = \sum\frac{2}{n^2}$ is conV. According to Limit Comp. Test, $\sum\frac{2n}{n^3+8}$ is conV.

Therefore, $\sum(-1)^n\cdot\frac{2n}{n^3+8}$ is ABS Convergent.

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wwHints and Remarks:

ww3: $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, $\cos 4\pi = 1$, ..., $\cos n\pi = (-1)^n$.

ww5.1: Compare $\frac{|\sin(2n)|}{n^2}$ with $\frac{1}{n^2}$ using Comparison Test based on $|\sin(x)| \le 1$.

☆ ww4: $(8n)! = 1\times2\times3\times\cdots\times(8n-4)\times(8n-3)\times(8n-2)\times(8n-1)\times 8n)$

$[8(n+1)]! = (8n+8)! = 1\times2\times3\times\cdots\times(8n-1)\times(8n)\times(8n+1)\times(8n+2)\times(8n+3)\times(8n+4)\times(8n+5)\times(8n+6)\times(8n+7)\times(8n+8)$

Spring 15 ※ 5 (a). Test $\sum_{n=1}^{\infty}(-1)^n\left(\frac{5}{2+\sqrt{10}}\right)^n$ for convergence.

Remark: In the solution file, it is tested via Ratio Test or Alternating Series Test. Actually, this is a Geometric Series, where $a_n = (-1)^n\cdot\left(\frac{5}{2+\sqrt{10}}\right)^n = \left(\frac{-5}{2+\sqrt{10}}\right)^n$ with $r = \frac{-5}{2+\sqrt{10}}$.

$|r| < 1$ directly implies it is conV and (ABS conV)